Scaling limit of vicious walks and two-matrix model

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We consider the diffusion scaling limit of the one-dimensional vicious walker model of Fisher and derive a system of nonintersecting Brownian motions. The spatial distribution of N particles is studied and it is described by use of the probability density function of eigenvalues of $N\times N$ Gaussian random matrices. The particle distribution depends on the ratio of the observation time t and the time interval T in which the nonintersecting condition is imposed. As t/T is going on from 0 to 1, there occurs a transition of distribution, which is identified with the transition observed in the two-matrix model of Pandey and Mehta. Despite of the absence of matrix structure in the original vicious walker model, in the diffusion scaling limit, accumulation of contact repulsive interactions realizes the correlated distribution of eigenvalues in the multimatrix model as the particle distribution.

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I. INTRODUCTION

The vicious walker models, in which random walkers walk without intersecting with any others in a given time interval, were introduced by Michael Fisher and applications of the models to various wetting and melting phenomena were described in his Boltzmann medal lecture [1]. Recently, using the standard one-to-one correspondence between walks and Young tableaux, Guttmann *et al.* [2] and Krattenthaler *et al.* [3] showed that exact formulas for total numbers of one-dimensional vicious walks, some of which were conjectured in previous papers [1,4–6], are derived following the theory of symmetric functions associated with Young diagrams [7–9] or the representation theory of classical groups [10]. Important analogies between the ensembles of Young tableaux and those of Gaussian random matrices were reported by Johansson [11], and then Baik [12] and Nagao and Forrester [13,14] studied the vicious walker models using the random matrix theory [15,16].

The purpose of the present paper is to demonstrate more explicit relations among the vicious walker model, the symmetric function called the Schur function and the Gaussian ensembles of random matrices by considering the diffusion scaling limit of the one-dimensional vicious walks. Since each random walk converges to a Brownian motion in the scaling limit, the limit process of N vicious walkers will be a system of *N nonintersecting Brownian motions* [17]. In order to enumerate all possible nonintersecting paths of walkers realized on a spatiotemporal plane, we use the so-called Lindström-Gessel-Viennot formula [18–20], which leads us to a useful determinantal expression for the transition probability density of nonintersecting Brownian motions. We found that its initial-configuration dependence can be generally described by using the Schur function and the well-known properties of this function enable us to define the nonintersecting Brownian motions in which *all particles start from a single position*. Because the nonintersecting condition will be imposed for a given time interval, say T, all the particles are immediately disunited from the initial point, and then they walk randomly keeping the nonintersecting condition. We have studied the time dependence of the spatial distribution of particle positions. We report in this paper that the position distribution of N nonintersecting Brownian motions can be identified with the distribution of eigenvalues of $N\times N$ complex Hermitian matrix H coupled to a real symmetric matrix A, in which H and A are randomly chosen from the Gaussian ensembles. Such a *two-matrix model* was studied by Pandey and Mehta [21,22], in which one parameter was introduced to control the coupling strength between two matrices. We will show that the time dependence of our process can be expressed by the parameter dependence of Pandey-Mehta's two-matrix model.

Here we consider the probability density function of N real variables $\{x_1,\cdots,x_N\}$ with a real parameter $\beta\geqslant 0$,

$$P_\beta(x_1,\cdots,x_N)=Ce^{-\beta\Sigma x_j^2/2}\prod_{1\leqslant j<k\leqslant N}|x_j-x_k|^\beta =C\exp[-\beta W(\{x_j\})], \tag{1}$$

with

$$W(\{x_j\})=\frac{1}{2}\sum_{j=1}^N x_j^2-\sum_{1\leqslant j<k\leqslant N}\ln|x_j-x_k|, \tag{2}$$

where C is a normalization constant. It is known that Eq. (1) with $\beta=1$, 2, and 4 describe the distributions of eigenvalues of random matrices in the Gaussian orthogonal, unitary, and symplectic ensembles, respectively (abbreviated as GOE, GUE, and GSE, respectively) [15]. For the one-dimensional

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N Brownian motions, in which all particles start from the origin and the nonintersecting condition is imposed in the time interval $(0,T]$, we will show that (i) at the very early stage, i.e., $t/T \ll 1$, the particle distribution is described by using GUE, (ii) as time t is going on, a transition from GUE to GOE is observed, and (iii) at the final stage $t=T$ the particle distribution can be identified with GOE. As shown by the second equality of Eq. (1), the Gaussian ensemble of random matrices can be regarded as the thermodynamical equilibrium of one-dimensional gas system with (two-dimensional) Coulomb repulsive potential (2) at the inverse temperature β. Here it should be noted that the vicious walkers on a lattice have only contact repulsive interactions to satisfy the nonintersecting condition. The global effective interactions among walkers are accumulated by taking the diffusion scaling limit and as its result a long-ranged Coulomb gas system is constructed. Such *emergence of long-range effects in macroscopic scales from systems having only short-ranged microscopic interactions* is found only at critical points in thermodynamical equilibrium systems, but it is a typical phenomenon observed in a various interacting particle systems in far from equilibrium.

In particular, in the limit $T\to\infty$, that is, when the nonintersecting condition will be imposed forever, we can derive a system of stochastic differential equations for the process with the drift terms that act as the repulsive two-body forces proportional to the inverse of distances between particles. In other words, the scaling limit of vicious walks with $T\to\infty$ can realize Dyson's Brownian motion model at $\beta=2$ [23]. It is reasonable to obtain such a stochastic process from the vicious walker model, since it is known that Dyson's Brownian motion model at $\beta=2$ can be mapped to the free fermion model [17,24]. The transition from GUE to GOE is, however, first reported for vicious walkers with $T<\infty$ and explained using the two-matrix model in the present paper.

II. MODEL AND LINDSTRÖM-GESSEL-VIENNOT DETERMINANT

One-dimensional vicious walks are defined as a subset of simple random walks as follows. Let $\{R_k^{s_j}\}_{k\geq 0}, j=1,2,\ldots,N$, be the N independent symmetric simple random walks on $\mathbf{Z}=\{\cdots,-2,-1,0,1,2\cdots\}$ started from N distinct positions, $2s_1<2s_2<\cdots<2s_N$, $s_j\in\mathbf{Z}$. That is,

$$R_0^{s_j}=2s_j, \quad \text{and} \quad R_{k+1}^{s_j}=R_k^{s_j}-1 \quad \text{or} \quad R_k^{s_j}+1,$$

for $j=1,2,\ldots,N, k=0,1,2,\ldots$. Fix the time interval K as a positive even number. The total number of walks is 2^{NK}, all of which are assumed to be realized with equal probability 2^{-NK}. We consider a subset of walks such that any of walkers does not meet other walkers up to time K. In other words, the condition

$$R_k^{s_1}<R_k^{s_2}<\cdots<R_k^{s_N}, \quad k=1,2,\ldots,K \tag{3}$$

is imposed. Such a subset of walks is called the vicious walks (up to time K) [1,4]. Let $N_N(K;\{e_j\}|\{s_j\})$ be the total number of the vicious walks, in which the N walkers starting from $2s_1<2s_2<\cdots<2s_N$ arrive at the positions $2e_1<2e_2<\cdots<2e_N$ at time K. Then the probability that such vicious walks with those fixed end points are realized in all possible random walks started from the given initial positions, which is denoted as $V_N(\{R_k^{s_j}\}_{k=0}^K;R_K^{s_j}=2e_j)$, is

$$V_N(\{R_k^{s_j}\}_{k=0}^K;R_K^{s_j}=2e_j)=\frac{N_N(K;\{e_j\}|\{s_j\})}{2^{NK}}.$$

We also consider the probability

$$V_N(\{R_k^{s_j}\}_{k=0}^K)=\sum_{e_1<e_2<\cdots<e_N} V_N(\{R_k^{s_j}\}_{k=0}^K;R_K^{s_j}=2e_j).$$

Consider a subset of the square lattice $\mathbf{Z}^2$,

$$\mathcal{L}_K=\{(x,y)\in\mathbf{Z}^2: x+y=\text{even}, \quad 0\leq y\leq K\},$$

and the set $\mathcal{E}_K$ of all edges that connect the nearest-neighbor pairs of vertices in $\mathcal{L}_K$. The lattice $(\mathcal{L}_K,\mathcal{E}_K)$ provides the spatiotemporal plane and each walk of the jth walker, $j=1,2,\cdots,N$, can be represented as a sequence of successive edges connecting vertices $\mathbf{S}_j=(2s_j,0)$ and $\mathbf{E}_j=(2e_j,K)$ on it, which we call the *lattice path* running from $\mathbf{S}_j$ to $\mathbf{E}_j$. If such lattice paths share a common vertex, they are said to intersect. Under the vicious walk condition (3), what we consider is a set of all N tuples of *nonintersecting paths* [20]. Let $\pi(\mathbf{S}\to\mathbf{E})$ be the set of all lattice paths from $\mathbf{S}$ to $\mathbf{E}$, and $\pi_0(\{\mathbf{S}_j\}_{j=1}^N\to\{\mathbf{E}_j\}_{j=1}^N)$ be the set of all N-tuples $(\pi_1,\cdots,\pi_N)$ of nonintersecting lattice paths, in which π_j runs from $\mathbf{S}_j$ to $\mathbf{E}_j$, $j=1,2,\cdots,N$. If we write the number of elements in a set A as $|A|$, then $N_N(K;\{e_j\}|\{s_j\})=|\pi_0(\{\mathbf{S}_j\}_{j=1}^N\to\{\mathbf{E}_j\}_{j=1}^N)|$.

The Lindström-Gessel-Viennot theorem gives [18–20] (see also [1,6,14]),

$$N_N(K;\{e_j\}|\{s_j\})=\det_{1\leq j,k\leq N}(|\pi(\mathbf{S}_k\to\mathbf{E}_j)|).$$

Since $|\pi(\mathbf{S}_k\to\mathbf{E}_j)|=\binom{K}{K/2+s_k-e_j}$, we have the following binomial determinantal expressions:

$$V_N(\{R_k^{s_j}\}_{k=0}^K;R_K^{s_j}=2e_j) =2^{-NK}\det_{1\leq j,k\leq N}\left(\binom{K}{K/2+s_k-e_j}\right) \tag{4}$$

and

$$V_N(\{R_k^{s_j}\}_{k=0}^K) =2^{-NK}\sum_{e_1<e_2<\cdots<e_N}\det_{1\leq j,k\leq N}\left(\binom{K}{K/2+s_k-e_j}\right). \tag{5}$$

III. SCALING LIMIT OF VICIOUS WALKS

Recently Krattenthaler *et al.* [3] evaluated the asymptotes of Eq. (5) for large K in the two special initial configurations, (i) $s_j=j-1$ and (ii) $s_j=2(j-1)$, as

$$V_N(\{R_k^{s_j}\}_{k=0}^K)=a_N b_N(\{s_j\})K^{-N(N-1)/4}[1+O(1/K)], \tag{6}$$

where

$$a_N=\begin{cases}(2^N/\pi)^{N/4}\prod_{j=1}^{N/2}(2j-2)! & \text{if} \quad N=\text{even}\\ (2^{N+1}/\pi)^{(N-1)/4}\prod_{j=1}^{(N-1)/2}(2j-1)! & \text{if} \quad N=\text{odd},\end{cases} \tag{7}$$

and

$$b_N(\{j-1\})=1, \quad b_N[\{2(j-1)\}]=2^{N(N-1)/2}. \tag{8}$$

We found that their result can be immediately generalized as

$$b_N[\{s(j-1)\}]=s^{N(N-1)/2}$$

for $s_j=s(j-1), s=1,2,3,\cdots$. This observation suggests that we can take the scaling limit $L\to\infty$, where the time interval $K\propto L$ and the initial spacing of walkers $s\propto\sqrt{L}$.

A. Schur function

In order to describe the scaling limit of the vicious walks, the symmetric function called the Schur function is useful. Here we give some of the fundamental properties of Schur function [7–10], which will be used below.

A partition $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_N)$ is a nonincreasing series of nonnegative integers, $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_N\geq 0$. Let V be the N-dimensional complex vector space. Then the Schur function $s_\lambda(z_1,\cdots,z_N)$ associated with λ is a function of $(z_1,\cdots,z_N)\in V$ defined by

$$s_\lambda(z_1,\cdots,z_N)=\frac{\det_{1\leq j,k\leq N}(z_j^{\lambda_k+N-k})}{\det_{1\leq j,k\leq N}(z_j^{N-k})}. \tag{9}$$

Let $\Delta_\lambda(\{z_j\})$ be the numerator of Eq. (9), which is an $N\times N$ determinant. If we set $z_{l_1}=z_{l_2}$ for $1\leq l_1<l_2\leq N$, then $\Delta_\lambda(\{z_j\})=0$, since the l_1th row is equal to the l_2th row. Then it is divisible by each of the differences $z_{l_1}-z_{l_2}, 1\leq l_1<l_2\leq N$, and hence by their product $\Pi_{1\leq j<k\leq N}(z_j-z_k)$. This product of all differences is known as the Vandermonde determinant, which is nothing but the denominator of Eq. (9);

$$\Delta_0(\{z_j\})\equiv\det_{1\leq j,k\leq N}(z_j^{N-k})=\prod_{1\leq j<k\leq N}(z_j-z_k). \tag{10}$$

Therefore it is concluded that the ratio of two determinants Δ_λ/Δ_0 is a polynomial in $z_1,\cdots,z_N$. Moreover, it can be readily seen from Eq. (9) that the Schur function is a homogeneous polynomial of degree $\Sigma_{j=1}^N\lambda_j$ in $z_1,\cdots,z_N$.

Let q be a complex variable and set $z_j=q^{j-1}$ in Eq. (9). Then we have

$$s_\lambda(1,q,q^2,\cdots,q^{N-1})=\frac{\det_{1\leq j,k\leq N}(q^{(j-1)(\lambda_k+N-k)})}{\det_{1\leq j,k\leq N}(q^{(j-1)(N-k)})}.$$

Appropriate application of the formula of Vandermonde determinant (10) gives the product form

$$s_\lambda(1,q,q^2,\ldots,q^{N-1})=q^{\Sigma(j-1)\lambda_j}\prod_{1\leq j<k\leq N}\frac{q^{\lambda_j-\lambda_k+k-j}-1}{q^{k-j}-1}.$$

Taking the limit $q\to 1$, we have the formula

$$s_\lambda(1,1,\ldots,1)=\prod_{1\leq j<k\leq N}\frac{\lambda_j-\lambda_k+k-j}{k-j}. \tag{11}$$

The Schur function is a character of the irreducible representation specified by λ of the group $GL(V)$ and Eq. (11) gives the dimension of the representation.

B. Diffusion scaling limit

We set

$$K=Lt, \quad s_j=\frac{\sqrt{L}}{2}x_j, \quad e_j=\frac{\sqrt{L}}{2}y_j, \tag{12}$$

for $j=1,2,\cdots,N$, and take the limit $L\to\infty$. Since in this limit each random walk $R_k^{s_j}$ converges to a Brownian motion, whose distribution function solves the diffusion equation, this scaling limit is especially called *diffusion scaling limit*. First we remark that, for each strictly increasing series of integers $y_1<y_2<\cdots<y_N$, a weakly decreasing series of integers $\xi(y)=(\xi_1(y),\cdots,\xi_N(y))$ can be assigned by setting

$$\xi_j(y)=y_{N-j+1}-(N-j), \quad j=1,2,\ldots,N. \tag{13}$$

Then we can prove that, for given $t>0$, $x_1<x_2<\cdots<x_N$, and $y_1<y_2<\cdots<y_N$,

$$\begin{aligned}\lim_{L\to\infty}\left(\frac{\sqrt{L}}{2}\right)^N & V_N(\{R_k^{\sqrt{L}x_j/2}\}_{k=0}^{Lt};R_{Lt}^{\sqrt{L}x_j/2}=\sqrt{L}y_j)\\ &=(2\pi t)^{-N/2}\det_{1\leq j,k\leq N}\left[\exp\left(-\frac{1}{2t}(x_k-y_j)^2\right)\right]\\ &=(2\pi t)^{-N/2}s_{\xi(y)}(e^{x_1/t},e^{x_2/t},\ldots,e^{x_N/t})\\ &\quad\times\exp\left(-\frac{1}{2t}\sum_{j=1}^N(x_j^2+y_j^2)\right)h_N(\{e^{x_j/t}\}),\end{aligned} \tag{14}$$

where $s_{\xi(y)}(z_1,\cdots,z_N)$ is the Schur function associated with $\xi(y)$, defined by Eq. (9) with $\lambda=\xi(y)$, and

$$\begin{aligned}h_N(\{z_j\})&\equiv\det_{1\leq j,k\leq N}(z_j^{k-1})=(-1)^{N(N-1)/2}\Delta_0(\{z_j\})\\ &=\prod_{1\leq j<k\leq N}(z_k-z_j).\end{aligned} \tag{15}$$

The proof is given as follows. Setting Eq. (12), we apply Stirling's formula to the right-hand side (RHS) of Eq. (4) multiplied by $(\sqrt{L}/2)^N$,

$$\lim_{L\to\infty} 2^{-NLt}\left(\frac{\sqrt{L}}{2}\right)^N \det_{1\le j,k\le N}\left(\binom{Lt}{\frac{Lt}{2}+\frac{\sqrt{L}(x_k-y_j)}{2}}\right)$$

$$=\det_{1\le j,k\le N}\left[\lim_{L\to\infty} 2^{-Lt}\left(\frac{\sqrt{L}}{2}\right)\binom{Lt}{\frac{Lt}{2}+\frac{\sqrt{L}(x_k-y_j)}{2}}\right]$$

$$=\det_{1\le j,k\le N}\left[\frac{1}{\sqrt{2\pi t}}e^{-(x_k-y_j)^2/2t}\right], \tag{16}$$

which gives the first equality of Eq. (14). For the second equality, we rewrite Eq. (16) as

$$(2\pi t)^{-N/2}e^{-\Sigma(x_j^2+y_j^2)/2t}\det_{1\le j,k\le N}(e^{x_k y_j/t}).$$

The determinant is written as

$$\det_{1\le j,k\le N}(e^{x_k y_j/t})=\frac{\det_{1\le j,k\le N}((e^{x_j/t})^{y_{N-k+1}})}{\det_{1\le j,k\le N}((e^{x_j/t})^{N-k})}\times h_N(\{e^{x_j/t}\}).$$

Using Eq. (13) and the definition of Schur function (9), the second equality of Eq. (14) is obtained.

We consider the rescaled one-dimensional lattice $\mathbf{Z}/(\sqrt{L}/2)$, where the unit length is $2/\sqrt{L}$, and let $\widetilde{R}_k^x$ denote the symmetric simple random walk starting from x on $\mathbf{Z}/(\sqrt{L}/2)$. Then Eq. (14) implies that

$$\lim_{L\to\infty} V_N(\{\widetilde{R}_k^{x_j}\}_{k=0}^{Lt};\widetilde{R}_{Lt}^{x_j}\in[y_j,y_j+dy_j])=f_N(t;\{y_j\}|\{x_j\})d^N y.$$

Here we can give two expressions for $f_N(t;\{y_j\}|\{x_j\})$,

$$\begin{aligned}f_N(t;\{y_j\}|\{x_j\})&=(2\pi t)^{-N/2}\times\det_{1\le j,k\le N}\left[\exp\left(-\frac{1}{2t}(x_k-y_j)^2\right)\right]\\&=(2\pi t)^{-N/2}s_{\xi(y)}(e^{x_1/t},e^{x_2/t},\dots,e^{x_N/t})\\&\quad\times\exp\left(-\frac{1}{2t}\sum_{j=1}^N(x_j^2+y_j^2)\right)h_N(\{e^{x_j/t}\}).\end{aligned} \tag{17}$$

Since the vicious walkers are defined by imposing the nonintersecting condition (3) up to a given time K, the process depends on the choice of K. That is, the process is *temporally inhomogeneous*. This feature should be inherited in the process obtained in the diffusion scaling limit. Since each random walk converges to a Brownian motion in the diffusion scaling limit, the limit process of the N vicious walkers can be called the N *nonintersecting Brownian motions*. Let $T>0$ and we consider the N nonintersecting Brownian motions in the time interval $(0,T]$. Set

$$\mathcal{N}_N(t;\{x_j\})=\int_{y_1<\dots<y_N} d^N y f_N(t;\{y_j\}|\{x_j\}).$$

For $0\le s<t\le T$, $x_1<\dots<x_N, y_1<\dots<y_N$, the transition probability density from the configuration $\{x_j\}$ at time s to $\{y_j\}$ at t is given by

$$g_N^T(s,\{x_j\};t,\{y_j\})=\frac{f_N(t-s;\{y_j\}|\{x_j\})\mathcal{N}_N(T-t;\{y_j\})}{\mathcal{N}_N(T-s;\{x_j\})}, \tag{18}$$

since the numerator in RHS gives the nonintersecting probability for $(0,T]$ specified with the configurations $\{x_j\}$ and $\{y_j\}$ at times s and t, respectively, and the denominator gives the probability only specified with $\{x_j\}$ at s, where we have used the Markov property of the process. The temporal inhomogeneity is obvious, since RHS depends not only $t-s$ but also $T-s$ and $T-t$.

C. $t\to\infty$ asymptote of $\mathcal{N}_N(t;\{x_j\})$

It should be noted that, since $\mathcal{N}_N(t;\{x_j\})$ is the integral of $f_N(t;\{y_j\}|\{x_j\})$ over all possible end positions $\{y_j\}$, it is the probability that N Brownian motions starting from $\{x_j\}$ do not intersect up to time t. Before studying the stochastic process defined by the transition probability density (18), here we assume $|\mathbf{x}|\equiv\Sigma_{j=1}^N|x_j|<\infty$ and evaluate the $t\to\infty$ asymptote of $\mathcal{N}_N(t;\{x_j\})$. In order to do that, the second expression of $f_N(t;\{y_j\}|\{x_j\})$ in (17) will be useful,

$$\mathcal{N}_N(t;\{x_j\})=\frac{e^{-\Sigma x_j^2/2t}}{(2\pi t)^{N/2}}h_N(\{e^{x_j/t}\})\int_{y_1<\dots<y_N}d^N y\times s_{\xi(y)}(e^{x_1/t},\dots,e^{x_N/t})e^{-\Sigma y_j^2/2t}.$$

By Eq. (11), (13), and (15),

$$\lim_{t\to\infty}s_{\xi(y)}(e^{x_1/t},\dots,e^{x_N/t})=s_{\xi(y)}(1,1,\dots,1)=h_N(\{y_j\})/\prod_{1\le j<k\le N}(k-j)$$

and

$$\lim_{t\to\infty}t^{N(N-1)/2}h_N(\{e^{x_j/t}\})=h_N(\{x_j\}).$$

We define

$$b_N(\{x_j\})=h_N(\{x_j\})/\prod_{1\le j<k\le N}(k-j).$$

Note that this definition of $b_N(\{x\})$ is consistent with Eq. (8). Then we have

$$\mathcal{N}_N(t;\{x_j\})=t^{-N^2/2}\frac{b_N(\{x_j\})}{(2\pi)^{N/2}N!}\int d^N y \;\; e^{-\Sigma y_j^2/2t}|h_N(\{y_j\})|\times[1+O(1/t)]$$
$$=t^{-N(N-1)/4}\frac{b_N(\{x_j\})}{(2\pi)^{N/2}N!}\int d^N u e^{-\Sigma u_j^2/2}|h_N(\{u_j\})|\times[1+O(1/t)],$$

as t tends to infinity, where we have used the facts that with the absolute values the product of differences $|h_N(\{y_j\})|$ is invariant under permutation of y_j, and $u_j=y_j/\sqrt{t}$. The last integral is the special case ($\gamma=1/2$ and $a=1/2$) of

$$\int d^N u \;\; e^{-a\Sigma u_j^2}\prod_{1\le j<k\le N}|u_k-u_j|^{2\gamma}=(2\pi)^{N/2}(2a)^{-N[\gamma(N-1)+1]/2}\prod_{j=1}^{N}\frac{\Gamma(1+j\gamma)}{\Gamma(1+\gamma)}, \tag{19}$$

which is found in Mehta [15] [Eq. (17.6.7) on page 354], whose proof was given in [25] by use of Selberg's integral [26]. Here $\Gamma(x)$ is the Gamma function with the values $\Pi_{j=1}^{N}\Gamma(1+j/2)=2^{-N(N-1)/2}(\sqrt{\pi}/2)^N N!a_N$ and $\Gamma(3/2)=\sqrt{\pi}/2$ and, where a_N is given by Eq. (7). Then we have

$$\mathcal{N}_N(t;\{x_j\})=t^{-\psi_N}2^{-2\psi_N}a_N b_N(\{x_j\})[1+O(1/t)] \tag{20}$$

with

$$\psi_N=\frac{1}{4}N(N-1), \tag{21}$$

as t tends to infinity, where ψ_N is known as the critical exponent of survival probability of vicious walkers [1,4,27,28]. Since

$$t^{-\psi_N}2^{-2\psi_N}b_N(\{x_j\})=(Lt)^{-\psi_N}b_N(\{\sqrt{L}x_j/2\}),$$

(20) suggests that the result (6) with (7) and (8) of Krattenthaler *et al.* shall be generalized for arbitrary initial positions of vicious walkers on the lattice.

IV. GAUSSIAN RANDOM MATRIX ENSEMBLES AND DYSON'S BROWNIAN MOTIONS

In this section we study two special choices of T; $T=t$ and $T\to\infty$. We show that there is an interesting correspondence between these choices of T and the Gaussian ensembles of random matrices. In order to see it we consider the limit $|\mathbf{x}|\to 0$, where $|\mathbf{x}|\equiv\Sigma_{j=1}^{N}|x_j|$. It will be shown that the second expression of $f_N(t;\{y_j\}|\{x_j\})$ in Eq. (17) is useful for taking this limit.

A. $T=t$ case and GOE

Since the first expression in Eq. (17) gives $\lim_{t\to 0}f_N(t;\{y_j\}|\{x_j\})=\Pi_{j=1}^{N}\delta(x_j-y_j)$ with Dirac's δ functions, $\mathcal{N}_N(0;\{x_j\})=1$ for any $\{x_j\}$. Then setting $T=t$ makes (18) depend only on $t-s$. Set $s=0$ and use the second expression in Eq. (17) for $f_N(t;\{y_j\}|\{x_j\})$ and $\mathcal{N}_N(t;\{x_j\})$. By virtue of the Schur function (11), for $t>0$ and $|\mathbf{x}|\ll 1$, we have

$$f_N(t;\{y_j\}|\{x_j\})=(2\pi t)^{-N/2}h_N(\{e^{x_j/t}\})s_{\xi(y)}(1,\ldots,1)\times e^{-\Sigma y_j^2/2t}[1+O(|x|)]$$
$$=\frac{t^{-N/2}}{(2\pi)^{N/2}}e^{-\Sigma y_j^2/2t}h_N(\{y_j\})\times\prod_{1\le j<k\le N}\frac{e^{x_k/t}-e^{x_j/t}}{k-j}\times[1+O(|x|)]$$

and

$$\mathcal{N}_N(t;\{x_j\})=(2\pi t)^{-N/2}h_N(\{e^{x_j/t}\})\int_{y_1<\cdots<y_N}d^N y\times s_{\xi(y)}(1,\ldots,1)e^{-\Sigma y_j^2/2t}[1+O(|x|)]$$
$$=\frac{t^{N(N-1)/4}}{(2\pi)^{N/2}c_N}\prod_{1\le j<k\le N}\frac{e^{x_k/t}-e^{x_j/t}}{k-j}[1+O(|x|)],$$

where the integral (19) was used and

$$c_N=\frac{2^{N(N-2)/2}}{\pi^{N/2}a_N}=\left(2^{N/2}\prod_{j=1}^{N}\Gamma(j/2)\right)^{-1}.$$

Then Eq. (18) gives

$$g_N^t(0,\{0\};t,\{y_j\})=c_N t^{-\zeta_N}e^{-\Sigma y_j^2/2t}h_N(\{y_j\})$$

for $y_1<\cdots<y_N$ with

$$\zeta_N=\frac{1}{4}N(N+1).$$

It means that

$$g_N^t(0,\{0\};t,\{y_j\})=N!g_N^{\rm GOE}(\{y_i\};t)$$

for $y_1<\cdots<y_N$, where

$$g_N^{\rm GOE}(\{y_j\};\sigma^2)=\frac{c_N}{N!}\sigma^{-2\zeta_N}\exp\left(-\frac{1}{2\sigma^2}\sum_{j=1}^{N}y_j^2\right)h_N(\{y_j\}) \tag{22}$$

is the probability density function of eigenvalues $\{y_j\}$ of random matrices in the Gaussian orthogonal ensemble with variance σ^2 [15].

B. $T\to\infty$ limit and GUE

Let

$$p_N(s,\{x_j\};t,\{y_j\}) \equiv \lim_{T\to\infty} g_N^T(s,\{x_j\};t,\{y_j\}).$$

By use of Eq. (20) we can determine the explicit form for any initial configuration $x_1<x_2<\cdots<x_N$ in this case as

$$p_N(0,\{x_j\};t,\{y_j\}) = \frac{h_N(\{y_j\})}{h_N(\{x_j\})} f_N(t;\{y_j\}|\{x_j\}), \tag{23}$$

where h_N is given by Eq. (15). Moreover, if we take the limit $|\mathbf{x}|\to 0$, we have

$$p_N(0,\{0\};t,\{y_j\}) = c'_N t^{-\zeta'_N} e^{-\Sigma y_j^2/2t} h_N(\{y_j\})^2, \tag{24}$$

with

$$\zeta'_N = \frac{N^2}{2} \quad \text{and} \quad c'_N = \left((2\pi)^{N/2} \prod_{j=1}^{N} \Gamma(j) \right)^{-1}.$$

That is, we have the identity

$$p_N(0,\{0\};t,\{y_j\}) = N!\, g_N^{\mathrm{GUE}}(\{y_i\};t),$$

for $y_1<\cdots<y_N$, where

$$g_N^{\mathrm{GUE}}(\{y_j\};\sigma^2) = \frac{c'_N}{N!} \sigma^{-2\zeta'_N} \exp\left(-\frac{1}{2\sigma^2} \sum_{j=1}^{N} y_j^2 \right) h_N(\{y_j\})^2 \tag{25}$$

is the probability density function of eigenvalues $\{y_j\}$ of random matrices in the Gaussian unitary ensemble with variance σ^2 [15].

In the case $T\to\infty$, the nonintersecting condition will be imposed forever, while in the case $T=t$, there will be no condition in the future. *The distributions of particles at present depend on the condition in the future.*

By generalizing the calculation, which we did in the case $T=t$, for arbitrary T and comparing the result with Eq. (24), we have

$$\frac{g_N^T(0,\{0\};t,\{y_j\})}{p_N(0,\{0\};t,\{y_j\})} = \overline{c}_N T^{\psi_N} \frac{\mathcal{N}_N(T-t;\{y_j\})}{h_N(\{y_j\})} \tag{26}$$

for $y_1<\cdots<y_N$, with Eq. (21) and

$$\overline{c}_N = \frac{c_N}{c'_N} = \pi^{N/2} \prod_{j=1}^{N} \frac{\Gamma(j)}{\Gamma(j/2)}.$$

When $N=2$, we can consider the process of one variable $y=y_2-y_1$. In this case g_2^T and p_2 define the *Brownian meander* and the *Bessel process*, respectively, both of which are stochastic processes well studied in probability theory [29]. The equality (26) can be regarded as the multivariable generalization of Imhof's relation [30] between the Brownian meander and the Bessel process.

C. Dyson's Brownian motions

In the limit $T\to\infty$ we have obtained the compact expression (23) for any $x_1<\cdots<x_N$ and $y_1<\cdots<y_N$. In this section, we show that a system of stochastic differential equations can be explicitly derived for Eq. (23). Using it we will explain why we have the GUE distribution.

Let

$$E_k(\{x_j\}) = \sum_{j=1; j\neq k}^{N} \frac{1}{x_k - x_j} \quad \text{for} \quad k=1,2,\ldots N.$$

It is easy to verify that

$$E_k(\{x_j\}) = \frac{\partial}{\partial x_k} \ln h_N(\{x_j\}), \tag{27}$$

for $k=1,2,\ldots,N$, and

$$\sum_{k=1}^{N} \left[\frac{\partial}{\partial x_k} E_k(\{x_j\}) + [E_k(\{x_j\})]^2 \right] = 0. \tag{28}$$

Using these equalities, we can prove that $p_N(0,\{x_j\};t,\{y_j\})$ solves the equation

$$\frac{\partial}{\partial t} u(t;\{x_j\}) = \frac{1}{2} \Delta u(t;\{x_j\}) + \sum_{k=1}^{N} E_k(\{x_j\}) \frac{\partial}{\partial x_k} u(t;\{x_j\}), \tag{29}$$

where $\Delta = \Sigma_{k=1}^N \partial^2/\partial x_k^2$. The proof is the following. First we remark that the first expression in Eq. (17) states that f_N is a finite summation of the products of Gaussian kernels and thus it satisfies the diffusion equation [31]. Therefore,

$$\frac{\partial}{\partial t} p_N(0,\{x_j\};t,\{y_j\}) = \frac{1}{2} \frac{h_N(\{y_j\})}{h_N(\{x_j\})} \Delta f_N(t;\{y_j\}|\{x_j\}).$$

Then we can find that, if $\{E_k(\{x_j\})\}$ satisfy the equations

$$\sum_{k=1}^{N} E_k(\{x_j\}) \frac{1}{h_N(\{x_j\})} \frac{\partial}{\partial x_k} f_N(t;\{y_j\}|\{x_j\}) = -\sum_{k=1}^{N} \left\{ \frac{\partial}{\partial x_k} \frac{1}{h_N(\{x_j\})} \right\} \left\{ \frac{\partial}{\partial x_k} f_N(t;\{y_j\}|\{x_j\}) \right\} \tag{30}$$

and

$$\sum_{k=1}^{N} E_k(\{x_j\}) \frac{\partial}{\partial x_k} \frac{1}{h_N(\{x_j\})} = -\frac{1}{2} \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2} \frac{1}{h_N(\{x_j\})}, \tag{31}$$

Equation (29) holds with $u(t;\{x_j\}) = p_N(0,\{x_j\};t,\{y_j\})$. It is easy to see that Eq. (30) is satisfied if Eq. (27) holds for any $k=1,2,\ldots,N$. Moreover, using Eq. (27), we can reduce Eq. (31) to Eq. (28). Then the proof is completed.

The above result implies that the process defined in the limit $T\to\infty$ is the system of N particles with positions

$x_1(t), x_2(t), \cdots, x_N(t)$ at time t on the real axis, whose time evolution is governed by the stochastic differential equations

$$dx_k(t) = E_k(\{x_j(t)\})dt + dB_k(t), \tag{32}$$

$k=1,2,\cdots,N$, where $\{B_k(t)\}_{k=1}^N$ are the independent standard Brownian motions

$$B_j(0)=0, \quad \langle B_j(t)\rangle = 0,$$

$$\langle [B_j(t)-B_j(s)][B_k(t)-B_k(s)]\rangle = |t-s|\delta_{jk}$$

for any $t,s>0, j,k=1,2,\ldots,N$. Because of the scaling property of Brownian motion, $\sqrt{a}B_j(t)$ is equal to $B_j(at)$ in distribution for arbitrary $a>0$. Then, if we set $t=2t'$ and write $x_k(t)=\tilde{x}_k(t')$, Eq. (32) is the $\alpha=0, \beta=2$ case of the equations,

$$d\tilde{x}_k(t') = -\beta \frac{\partial}{\partial \tilde{x}_k} W^{\alpha}[\{\tilde{x}_j(t')\}]dt' + \sqrt{2}dB_k(t'), \tag{33}$$

$k=1,2,\ldots,N$, with

$$W^{\alpha}(\{\tilde{x}_j\}) = \alpha \sum_{j=1}^{n} \frac{\tilde{x}_j^2}{2} - \sum_{1\le j<k\le n} \ln(\tilde{x}_k - \tilde{x}_j).$$

When $\alpha=1$, Eq. (33) is known as the stochastic differential equations for the Dyson Brownian motions at the inverse temperature β and the stationary distribution $\propto \exp[-\beta W^1(\{\tilde{x}_j\})]$ [23]. If $\alpha=0$, the factor $\exp(-\beta\alpha\Sigma\tilde{x}_j^2/2)$ will be replaced by $\exp(-\Sigma\tilde{x}_j^2/4t')$ for finite t' and thus when $t'=\sigma^2/\beta$ we may have the Gaussian distribution $\propto \exp[-(\beta/4\sigma^2)\Sigma\tilde{x}_j^2]h_N(\{\tilde{x}_j\})^{\beta}$. Setting $\beta=2$ gives the form (25).

It should be noted that the system of diffusion equations describing the Dyson Brownian motions with $\beta=2$ can be mapped to the free fermion model [17,24].

For general $T<\infty$, we will have the stochastic differential equations

$$dx_k(t) = E_k^T(\{x_j(t)\})dt + dB_k(t),$$

for $k=1,2,\ldots,N$, with

$$E_k^T(\{x_j\}) = \frac{\partial}{\partial x_k} \ln \mathcal{N}_N(T-t;\{x_j\}).$$

V. TWO-MATRIX MODEL

In Sec. III we have constructed a system of nonintersecting Brownian motions in one dimension as the diffusion scaling limit of vicious walks. The obtained transition probability density (18) is temporally inhomogeneous and the particle distribution depends not only the observation time $t-s$ but also on the time interval T, in which nonintersecting condition is imposed. In the case that all particles start from the origin at time $s=0$, it was shown in Sec. IV that, (i) if $T=t$, it can be identified with the eigenvalue distribution of random matrices in GOE, and (ii) if $T\to\infty$, it becomes GUE.

For a fixed $T<\infty$, the above results are summarized as follows. Consider the one-dimensional N Brownian motions all starting from the origin at time $t=0$. We impose the nonintersecting condition for the time interval $(0,T]$. As the ratio $t/T\to 0$, the particle distribution is asymptotically described by GUE. On the other hand, at $t=T$, it can be identified with GOE. This implies that as time t is going on from 0 to T, there occurs a transition of distribution from GUE to GOE. In this section, we study this transition.

GUE is the ensemble of complex Hermitian matrices and GOE is that of real symmetric matrices. The degrees of freedom are, when the matrix sizes are N, N^2 and $N(N+1)/2$, respectively. If we change the variables from these independent matrix elements to the eigenvalues and other mutually independent variables, and then if we integrate the distribution functions over all variables other than eigenvalues, we will have the probability density functions for N real eigenvalues as Eqs. (25) and (22) [15].

Although the vicious walker model has no matrix structure at all, here we show that its diffusion scaling limit, nonintersecting Brownian motions, can be regarded as the reduction of a one-parameter family of ensembles of matrix structures to a variable space of eigenvalues. The "hidden structure" is not a single matrix but a two-matrix model, in which a complex Hermitian matrix is coupled with a real symmetric matrix.

In the first section we will derive the two-matrix model from the nonintersecting Brownian motions and the transition from GUE to GOE will be discussed in the second section. In the third section we will show that the obtained two-matrix model can be identified with the two-matrix model of Pandey and Mehta [21,22] by appropriate scale transformation of matrix elements.

A. From vicious walker model to two-matrix model

The generalized Imhof relation (26) with Eq. (24) gives

$$g_N^T(0,\{0\};t,\{y_j\}) \propto e^{-\Sigma y_j^2/2t} h_N(\{y_j\}) \int d^N z \, \mathrm{sgn}[h_N(\{z_j\})] \times \det_{1\le j,k\le N}\left[\exp\left(-\frac{1}{2(T-t)}(y_j-z_k)^2\right)\right], \tag{34}$$

where $\mathrm{sgn}(x)=x/|x|$. The RHS is rewritten as

$$h_N(\{y_j\}) \int d^N z \, \mathrm{sgn}[h_N(\{z_j\})] \det_{1\le j,k\le N}\left[\exp\left(-\frac{1}{2t}y_j^2 - \frac{1}{2(T-t)}(y_j-z_k)^2\right)\right]$$

$$= h_N(\{y_j\}) \int d^N z \;\; \mathrm{sgn}[h_N(\{z_j\})] e^{-\Sigma z_j^2/2T} \times \det_{1\le j,k\le N}\left[\exp\left(-\frac{T}{2t(T-t)}\left(y_j - \frac{t}{T}z_k\right)^2\right)\right].$$

Setting $(t/T)z_j = a_j, j=1,2,\ldots,N$, we have

$$g_N^T(0,\{0\};t,\{y_j\}) \propto h_N(\{y_j\}) \int d^N a \, \mathrm{sgn}[h_N(\{a_j\})] \times \exp\left(-\frac{T}{2t^2}\sum_{j=1}^N a_j^2\right) \times \det_{1\le j,k\le N}\left[\exp\left(-\frac{T}{2t(T-t)}(y_j-a_k)^2\right)\right]. \quad (35)$$

Consider an ensemble of $N\times N$ real symmetric matrices $\{A\}$ with an integration measure

$$dA \equiv \prod_{1\le j\le k\le N} dA_{jk}.$$

Let $\{a_1,a_2,\cdots,a_N\}$ be the eigenvalues of the matrix A and $\{p_1,p_2,\cdots,p_{N(N-1)/2}\}$ be other mutually independent variables. Then

$$dA = J(\{a_j\},\{p_j\})\prod_{j=1}^N da_j \prod_{k=1}^{N(N-1)/2} dp_k,$$

where $J(\{a_j\},\{p_j\})$ is the Jacobian

$$J(\{a_j\},\{p_j\}) = \left|\frac{\partial(A_{11},A_{12},\ldots,A_{NN})}{\partial(a_1,\ldots,a_N,p_1,\ldots,p_{N(N-1)/2})}\right|.$$

It is known that we can write

$$J(\{a_j\},\{p_k\}) = |h_N(\{a_j\})| f(\{p_k\}),$$

where $f(\{p_k\})$ is independent of a_j's [15]. Therefore, for any function $G(\{a_j\})$ of $\{a_1,\ldots,a_N\}$, we have the identity

$$\int dA \;\; G(\{a_j\}) = c\int \prod_{j=1}^N da_j |h_N(\{a_j\})| G(\{a_j\}) \quad (36)$$

with

$$c = \int \prod_{k=1}^{N(N-1)/2} dp_k f(\{p_j\}).$$

Set

$$G(\{a_j\}) = \frac{1}{|h_N(\{a_j\})|}\mathrm{sgn}[h_N(\{a_j\})]\exp\left(-\frac{T}{2t^2}\sum_{j=1}^N a_j^2\right) \times \det_{1\le j,k\le N}\left[\exp\left(-\frac{T}{2t(T-t)}(y_j-a_k)^2\right)\right].$$

Then using formula (36), (35) becomes

$$g_N^T(0,\{0\};t,\{y_j\}) \propto h_N(\{y_j\}) \int dA \frac{1}{h_N(\{a_j\})} \times \exp\left(-\frac{T}{2t^2}\sum_{j=1}^N a_j^2\right) \times \det_{1\le j,k\le N}\left[\exp\left(-\frac{T}{2t(T-t)}(y_j-a_k)^2\right)\right]. \quad (37)$$

Next we use the following integral formula [32–34]; for $N\times N$ Hermitian matrices A and B having eigenvalues $\{a_1,\ldots,a_N\}$ and $\{b_1,\ldots,b_N\}$, respectively, and for any constant γ,

$$\int dU \exp[\gamma\,\mathrm{tr}(A-U^\dagger BU)^2] \propto \frac{1}{h_N(\{a_j\})h_N(\{b_j\})} \times \det_{1\le j,k\le N}[\exp(\gamma(a_j-b_k)^2)],$$

where the integral is taken over the group of unitary matrices U. Then Eq. (37) can be written as

$$g_N^T(0,\{0\};t,\{y_j\}) \propto h_N(\{y_j\})^2 \int dU \int dA \exp\left(-\frac{T}{2t^2}\mathrm{tr}\,A^2\right) \times \exp\left(-\frac{T}{2t(T-t)}\mathrm{tr}(U^\dagger YU-A)^2\right), \quad (38)$$

where Y is the $N\times N$ diagonal matrix such that $Y_{jk}=y_j\delta_{jk}$. Since U is a unitary matrix, $H=U^\dagger YU$ is an $N\times N$ complex Hermitian matrix. Then the integrand of Eq. (38) can be regarded as a weight for two matrices H and A given as

$$\exp[-\mathrm{tr}(\gamma_H H^2 - \gamma_{HA}HA + \gamma_A A^2)]$$

with

$$\gamma_H = \frac{T}{2t(T-t)}, \quad \gamma_{HA} = \frac{T}{t(T-t)}, \quad \gamma_A = \frac{T^2}{2t^2(T-t)}. \quad (39)$$

Consider an ensemble of $N\times N$ complex Hermitian matrices $\{H\}$ with the integration measure

$$dH = \prod_{1\le j\le k\le N} d\mathrm{Re}(H_{jk}) \prod_{1\le j<k\le N} d\mathrm{Im}(H_{jk}).$$

For each complex Hermitian matrix H, let $\{y_1,\ldots,y_N\}$ be a set of eigenvalues and U be the $N\times N$ unitary matrix such that $H=U^\dagger YU$ with $Y_{jk}=y_j\delta_{jk}$. Then it is known that the integration measure dH can be factorized into the product of the Haar measure for unitary matrices dU and an integration measure for eigenvalues [15,16],

$$dH \propto dU \times h_N(\{y_j\})^2 \prod_{j=1}^N dy_j.$$

Now we introduce a two-matrix model, which consists of an $N\times N$ real symmetric matrix A and an $N\times N$ complex Hermitian matrix H, with a probability density function

$$\mu_N(H,A)=\frac{1}{Z_N}\exp(-\mathrm{tr}[\gamma_H H^2-\gamma_{HA}HA+\gamma_A A^2)].$$

Here $\gamma_H,\gamma_{HA},\gamma_A$ are given as Eq. (39) and Z_N is the partition function of the two-matrix model,

$$Z_N=\int dH\int dA\exp[-\mathrm{tr}(\gamma_H H^2-\gamma_{HA}HA+\gamma_A A^2)].$$

Then the relation

$$g_N^T(0,\{0\};t,\{y_j\})\propto h_N(\{y_j\})^2\int dU\int dA\,\mu_N(U^\dagger YU,A)$$

is established.

B. Transition from GUE to GOE

Consider the Gaussian ensembles of real symmetric matrices $\{A\}$ and complex Hermitian matrices $\{H\}$ with sizes N with the probability density functions

$$\nu_N(A)=C_A\exp\left(-\frac{1}{2\sigma_A^2}\mathrm{tr}\,A^2\right)$$

and

$$\tilde{\nu}_N(H)=C_H\exp\left(-\frac{1}{2\sigma_H^2}\mathrm{tr}\,H^2\right),$$

respectively, where

$$\sigma_A^2=\frac{t^2}{T},\quad \sigma_H^2=t\left(1-\frac{t}{T}\right),$$

and $C_A=2^{-N/2}(\pi\sigma_A^2)^{-\zeta_N}$, $C_H=2^{-N/2}(\pi\sigma_H^2)^{-\zeta_N'}$. Then consider the convolution

$$\hat{\mu}_N(H)=\int dA\,\nu_N(A)\tilde{\nu}_N(H-A).$$

Since, for $1\leqslant j,k\leqslant N$,

$$H_{jk}=\mathrm{Re}(H_{jk})+i\,\mathrm{Im}(H_{jk})$$

with $i=\sqrt{-1}$, and

$$\mathrm{Re}(A_{jk})=A_{jk},\quad \mathrm{Im}(A_{jk})=0,$$

the convolution is also Gaussian distribution in the form

$$\hat{\mu}_N(H)\propto\exp\left(-\sum_{j,k}\left\{\frac{[\mathrm{Re}(H_{jk})]^2}{2(\sigma_H^2+\sigma_A^2)}+\frac{[\mathrm{Im}(H_{jk})]^2}{2\sigma_H^2}\right\}\right).$$

Then Eq. (38) gives

$$g_N^T(0,\{0\};t,\{y_j\})\propto h_N(\{y_j\})^2\int dU\hat{\mu}_N(H)$$
$$\propto h_N(\{y_j\})^2\int dU\exp\left(-\sum_{j,k}\left\{\frac{[\mathrm{Re}(H_{jk})]^2}{2\sigma_{\mathrm{Re}}^2}+\frac{[\mathrm{Im}(H_{jk})]^2}{2\sigma_{\mathrm{Im}}^2}\right\}\right), \tag{40}$$

where $H=U^\dagger YU$ and

$$\sigma_{\mathrm{Re}}^2=t,\quad \sigma_{\mathrm{Im}}^2=t\left(1-\frac{t}{T}\right). \tag{41}$$

Now the transition from GUE to GOE is explicitly represented by the time-dependent variances (41). With a fixed finite T, if $0<t\ll T$, $\sigma_{\mathrm{Re}}^2=t\simeq\sigma_{\mathrm{Im}}^2$. Then the real and imaginary parts of complex Hermitian matrix elements are equally distributed as in GUE. While σ_{Re}^2 increases linearly in t, σ_{Im}^2 increases in time t only up to time $t=T/2$ and then decreases in time. At time $t=T$, $\sigma_{\mathrm{Im}}^2=0$, which implies that the imaginary parts of matrix elements are zeros with probability one. Then the distribution is identified with GOE.

C. Pandey-Mehta's two-matrix model

As an interpolation between GUE and GOE, Pandey and Mehta introduced a family of Gaussian ensembles of Hermitian matrices $\{H\}$ with one parameter $\alpha\in[0,1]$ [21,22],

$$\mu_N^{\mathrm{PM}}(H,\alpha)=C_{\mathrm{PM}}\exp\left(-\sum_{j,k}\left\{\frac{[\mathrm{Re}(H_{jk})]^2}{4v^2}+\frac{[\mathrm{Im}(H_{jk})]^2}{4v^2\alpha^2}\right\}\right), \tag{42}$$

where

$$v^2=\{2(1+\alpha^2)\}^{-1}$$

and

$$C_{\mathrm{PM}}=2^{-N/2}\alpha^{-N(N-1)/2}(2\pi v^2)^{-N^2/2}.$$

Set

$$\kappa=\sqrt{\frac{t(2T-t)}{T}}. \tag{43}$$

Then, it is easy to see that, if

$$\alpha^2=1-\frac{t}{T}, \tag{44}$$

the equality

$$\kappa^N\hat{\mu}_N(\kappa H)=\mu_N^{\mathrm{PM}}(H,\alpha) \tag{45}$$

is established.

For an even integer N and an antisymmetric $N\times N$ matrix $B=(b_{jk})$ we put

$$\mathrm{Pf}_{1\le j<k\le N}(b_{jk})=\frac{1}{(N/2)!}\sum_{\sigma}\mathrm{sgn}(\sigma)b_{\sigma(1)\sigma(2)}\times b_{\sigma(3)\sigma(4)}\dots b_{\sigma(N-1)\sigma(N)},$$

where the summation is extended over all permutations σ of $(1,2,\dots,N)$ with restriction $\sigma(2k-1)<\sigma(2k)$, $k=1,2,\dots,N/2$. This expression is known as the *Pfaffian* [20]. Pandey and Mehta showed that the probability density function of eigenvalues $\{y_j\}$ of the complex Hermitian matrices, which are distributed following Eq. (42), is given by

$$g_N^{\mathrm{PM}}(\{y_j\},\alpha)=C_N(\alpha)\exp\left[-\frac{1}{2}(1+\alpha^2)\sum_{j=1}^{N}y_j^2\right]\times h_N(\{y_j\})\mathrm{Pf}_{1\le j<k\le N}(F_{jk}), \tag{46}$$

where, setting

$$f(x)\equiv\left(\frac{1-\alpha^4}{\pi\alpha^2}\right)^{1/2}\int_0^x\exp\left(-\frac{1-\alpha^4}{4\alpha^2}y^2\right)dy,$$

if N is even, $F_{jk}=f(y_j-y_k), j,k=1,2,\dots,N$, and if N is odd, we use above and in addition $F_{j,N+1}=-F_{N+1,j}=1$, $j=1,2,\dots,N, F_{N+1,N+1}=0$, and $C_N(\alpha)^{-1}=2^{3N/2}(1-\alpha^2)^{N(N-1)/4}(1+\alpha^2)^{-N(N+1)/4}\prod_{j=1}^N\Gamma(1+j/2)$ [21,22]. Then the relation (40) and the equality (45) with (43) and (44) imply the expression

$$g_N^T(0,\{0\};t,\{y_j\})=c_NT^{N(N-1)/4}t^{-N^2/2}h_N(\{y_j\})\times\exp\left(-\frac{1}{2t}\sum_{j=1}^{N}y_j^2\right)\times\mathrm{Pf}_{1\le j<k\le\hat{N}}[\tilde{F}_{jk}(T-t,\{y_l\})], \tag{47}$$

where

$$\hat{N}=\begin{cases}N & \text{if } N \text{ is even}\\ N+1 & \text{if } N \text{ is odd,}\end{cases}$$

and

$$\tilde{F}_{jk}(t,\{y_l\})=\begin{cases}\dfrac{2}{\sqrt{\pi}}\mathrm{Erf}\left(\dfrac{y_k-y_j}{2\sqrt{t}}\right) & \text{if } 1\le j,k\le N\\ 1 & \text{if } 1\le j\le N, k=N+1\\ -1 & \text{if } j=N+1, 1\le k\le N\\ 0 & \text{if } j=k=N+1,\end{cases}$$

with

$$\mathrm{Erf}(x)=\int_0^x du\, e^{-u^2}.$$

In order to derive Eq. (46), Pandey and Mehta performed integration over alternative variables and then used the theory of Pfaffian [15,21,22]. In Appendix, we will give the *integration version* of Okada's minor-summation formula [35]. It should be noted that using it Eq. (47) can be readily obtained from Eq. (34) and this derivation provides another proof of the equality (45).

VI. CONCLUDING REMARKS

In the present paper we performed the diffusion scaling limit of vicious walker model in one dimension and constructed the nonintersecting Brownian motions for any finite number N of particles all starting from the origin. There the Schur function plays an important role to represent the transition probability density. We have shown that the spatial distribution of particles depends not only the observation time t but also on the time interval T in which the nonintersecting condition is imposed, and it can be described by use of the probability density function of eigenvalues of $N\times N$ random matrices in Gaussian ensembles. It was shown that the particle distribution depends on the ratio t/T and a transition from GUE distribution to GOE distribution occurs in its time development. Such a transition between different ensembles of random matrices had been studied in the two-matrix model by Pandey and Mehta, in which a parameter α was introduced and a one-parameter family of random matrix ensembles was considered. The present work showed that the scaling limit of vicious walk model realizes such a two-matrix model as a stochastic process, in such a way that the parameter α is continuously changed following Eq. (44) as the system is developing in time t up to T.

In the present paper, since we have considered only the transition probability density between two different times, a two-matrix model was analyzed. As briefly reported in [36], however, multitime correlations among particles at intermediate times between $t=0$ and $t=T$ can be identified with intermatrix correlations in a multimatrix model. The corresponding multimatrix model for calculating M-intermediate-time correlation functions is a version of Nagao's $(M+1)$ matrix model, where one real symmetric matrix is combined at the end of a chain of M complex Hermitian matrices [37]. This observation implies that the diffusion scaling limit of the vicious walker model is mathematically identified with a matrix chain, which is set along the time axis. The time development from GUE to GOE in the process can be then regarded as appearance of an *edge effect*, as the observation time t on the time axis is approaching to the end point $t=T$ in this chain structure. Further study on the relations between multimatrix models and nonequilibrium interacting particle systems is desired.

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APPENDIX: INTEGRATION VERSION OF OKADA'S MINOR-SUMMATION FORMULA

Assume $1 \leqslant n \leqslant m$ and let $z(j,k), 1 \leqslant j \leqslant n, 1 \leqslant k \leqslant m$, be indeterminates and $Z(z(j,k))$ be the $n \times m$ matrix with (j,k)-element $z(j,k)$. We consider the sum of all minors of Z with a given size r. That is, we define

$$d(a_1, \ldots, a_r) = \sum_{1 \leqslant b_1 < b_2 < \cdots < b_r \leqslant m} \det_{1 \leqslant j,k \leqslant r}[z(a_j, b_k)]$$

for $r = 1,2, \ldots, m$. Okada proved the following equalities known as the *minor-summation formula* [35]; if r is odd, then

$$d(a_1, \ldots, a_r) = \mathrm{Pf}\begin{pmatrix} 0 & d(a_1) & d(a_2) & \cdots & d(a_r) \\ -d(a_1) & 0 & d(a_1,a_2) & \cdots & d(a_1,a_r) \\ -d(a_2) & -d(a_1,a_2) & 0 & \cdots & d(a_2,a_r) \\ & & \cdots & & \\ -d(a_r) & -d(a_1,a_r) & -d(a_2,a_r) & \cdots & 0 \end{pmatrix};$$

if r is even, then

$$d(a_1, \cdots, a_r) = \mathrm{Pf}\begin{pmatrix} 0 & d(a_1,a_2) & d(a_1,a_3) & \cdots & d(a_1,a_r) \\ -d(a_1,a_2) & 0 & d(a_2,a_3) & \cdots & d(a_2,a_r) \\ -d(a_1,a_3) & -d(a_2,a_3) & 0 & \cdots & d(a_3,a_r) \\ & \cdots & & & \\ -d(a_1,a_r) & -d(a_2,a_r) & -d(a_3,a_r) & \cdots & 0 \end{pmatrix}.$$

Now we give the integration version of Okada's formula. Let $z(x,y)$ be a square integrable continuous function of real variables x,y. Then

$$\int_{-\infty < y_1 < \cdots < y_n < \infty} d^n y \det_{1 \leqslant j,k \leqslant n}[z(x_j, y_k)] = \mathrm{Pf}_{1 \leqslant j < k \leqslant \hat{n}}[F_{jk}(\{x_l\})], \tag{A1}$$

where

$$\hat{n} = \begin{cases} n & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd,} \end{cases}$$

and

$$F_{jk}(\{x_l\}) = \begin{cases} I_z(x_j, x_k) & \text{if } 1 \leqslant j < k \leqslant n \\ -I_z(x_j, x_k) & \text{if } 1 \leqslant k < j \leqslant n \\ I_z(x_j) & \text{if } 1 \leqslant j \leqslant n, k = n+1 \\ -I_z(x_k) & \text{if } j = n+1, 1 \leqslant k \leqslant n \\ 0 & \text{if } 1 \leqslant j = k \leqslant n+1, \end{cases}$$

with

$$I_z(x_j) = \int_{-\infty}^{\infty} z(x_j, y) dy,$$

$$I_z(x_j, x_k) = \int_{-\infty < y_1 < y_2 < \infty} \begin{vmatrix} z(x_j, y_1) & z(x_j, y_2) \\ z(x_k, y_1) & z(x_k, y_2) \end{vmatrix} dy_1 dy_2.$$

The proof is the following. We write the integral in left-hand side of Eq. (A1) as a limit of summation

$$\int_{-\infty < y_1 < \cdots < y_n < \infty} d^n y \det_{1 \leqslant j,k \leqslant n}[z(x_j, y_k)]$$
$$= \lim_{M \to \infty} \int_{-M/2 \leqslant y_1 < \cdots < y_n \leqslant M/2} d^n y \det_{1 \leqslant j,k \leqslant n}[z(x_j, y_k)]$$
$$= \lim_{M \to \infty} \lim_{\delta \to 0} \delta^n \sum_{1 \leqslant b_1 < \cdots < b_N \leqslant m(M,\delta)} \det_{1 \leqslant j,k \leqslant n}\{z[x_j, \hat{y}(b_k)]\},$$

where $m(M,\delta) = [M/\delta]$, the greatest integer not greater than M/δ, and

$$\hat{y}(b) = \frac{M-\delta}{m(M,\delta)-1} b - \left\{ \frac{M-\delta}{m(M,\delta)-1} + \frac{M}{2} \right\}.$$

Let

$$\tilde{d}(x_j) = \sum_{1 \leqslant b_1 \leqslant m(M,\delta)} z[x_j, \hat{y}(b_1)],$$

$$\tilde{d}(x_j, x_k) = \sum_{1 \leqslant b_1 < b_2 \leqslant m(M,\delta)} \begin{vmatrix} z[x_j, \hat{y}(b_1)] & z[x_j, \hat{y}(b_2)] \\ z[x_k, \hat{y}(b_1)] & z[x_k, \hat{y}(b_2)] \end{vmatrix},$$

and set

$$S_{jk}(\{x_l\}) = \begin{cases} \tilde{d}(x_j,x_k) & \text{if} \quad 1 \leqslant j < k \leqslant n \\ -\tilde{d}(x_j,x_k) & \text{if} \quad 1 \leqslant k < j \leqslant n \\ \tilde{d}(x_j) & \text{if} \quad 1 \leqslant j \leqslant n, k=n+1 \\ -\tilde{d}(x_j) & \text{if} \quad j=n+1, 1 \leqslant k \leqslant n \\ 0 & \text{if} \quad 1 \leqslant j=k \leqslant n+1. \end{cases}$$

Then Okada's formula gives

$$\int_{-\infty<y_1<\cdots<y_n<\infty} d^n y \det_{1\leqslant j,k\leqslant n}[z(x_j,y_k)] = \lim_{M\to\infty}\lim_{\delta\to 0} \delta^n \mathrm{Pf}_{1\leqslant j<k\leqslant \hat{n}}[S_{jk}(\{x_l\})]. \tag{A2}$$

Since the Pfaffian in Eq. (A2) is a finite summation of $n/2$ products of $\tilde{d}(x_j,x_k)$'s if n is even, and it is a finite summation of $(n-1)/2$ products of $\tilde{d}(x_j,x_k)$'s multiplied by $\tilde{d}(x_l)$ if n is odd, we may have

$$\int_{-\infty<y_1<\cdots<y_n<\infty} d^n y \det_{1\leqslant j,k\leqslant n}[z(x_j,y_k)] = \mathrm{Pf}_{1\leqslant j<k\leqslant \hat{n}}\left(\lim_{M\to\infty}\lim_{\delta\to 0} \delta^{\alpha(j,k)} S_{jk}(\{x_l\})\right),$$

where

$$\alpha(j,k)=2 \quad \text{for} \quad 1\leqslant j<k\leqslant n,$$

$$\alpha(j,n+1)=1 \quad \text{for} \quad 1\leqslant j\leqslant n.$$

Since $z(x,y)$ is assumed to be square integrable and continuous,

$$\lim_{M\to\infty}\lim_{\delta\to 0} \delta S_{jn+1}(\{x_l\}) = I_z(x_j),$$

$$\lim_{M\to\infty}\lim_{\delta\to 0} \delta^2 S_{jk}(\{x_l\}) = I_z(x_j,x_k),$$

for $1\leqslant j,k\leqslant n$. Then the proof is completed.

By elementary calculation we can show that

$$I_z(x)=1,$$

$$I_z(x,y)=\frac{2}{\sqrt{\pi}}\mathrm{Erf}\left(\frac{y-x}{2\sqrt{t}}\right)$$

for $z(x,y)=1/\sqrt{2\pi t}e^{-(x-y)^2/2t}$. Applying Eq. (A1) with $n=N$, the expression (47) is obtained from Eq. (34).

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